

Conservation laws for under determined systems of differential equations

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Abstract. This work extends the Ibragimov's conservation theorem for partial differential equations [*J. Math. Anal. Appl.* 333 (2007) 311-328] to under determined systems of differential equations. The concepts of adjoint equation and formal Lagrangian for a system of differential equations whose the number of equations is equal to or lower than the number of dependent variables are defined. It is proved that the system given by an equation and its adjoint is associated with a variational problem (with or without classical Lagrangian) and inherits all Lie-point and generalized symmetries from the original equation. Accordingly, a Noether theorem for conservation laws can be formulated.

Keywords: Adjoint equation, nonlinear differential equation; variational problem; Lagrangian; symmetry; conservation laws; Ginzburg-Landau system.

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INTRODUCTION

Conservation laws play a vital role in the study of partial differential equations (PDEs) in the search for qualitative properties such as integrability, stability, existence of global solutions and the linearizability conditions [1, 2, 3, 4, 5]. Their usefulness has considerably increased since the work by Jacobi [6] in 1884, who showed a connection between conserved quantities and symmetries of the equations of a particle's motion in classical mechanics. Klein [7] has obtained similar result for the equations of the general relativity and predicted that a connection between conservation laws and symmetries could be found for any differential equation obtained from a variational principle. Noether [8] has showed that the conservation laws were associated with the invariance of variational integrals with respect to continuous transformation groups. She obtained the sufficient condition for existence of conservation laws. In 1921, Bessel-Hagen [9] applied Noether's theorem with the so-called "divergence" condition to the Maxwell equations and calculated their conservation laws. In 1951, Hill [10] presented the explicit formula in terms of variations for conservation laws in the case of a first order Lagrangian. Ibragimov [11] proved the generalized version of Noether's theorem and conservation laws related to the invariance of the extremal values of variational integrals. He derived the

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necessary and sufficient condition for the existence of conservation laws and gave the explicit expressions in the case of a Lagrangian of any order.

Recently, some relevant results on conservation laws have emerged. See for instance the works by Wolf [12], Anco [13, 14], Poole [15], Ibragimov [16, 17], Naz [18] and Khamitova [19]. Despite all this progress, there remains an important question: How to associate a conservation law with every infinitesimal generator of symmetries of an arbitrary differential equation? Ibragimov [17] achieved this goal for any system of differential equations where the number of equations is equal to the number of dependent variables. Our work extends Ibragimov's result to under determined system of differential equations.

NOTATIONS, BASIC DEFINITIONS AND THEOREMS

Consider X , an n -dimensional independent variables space, and $U = \bigotimes_{j=1}^m U^j$, an m -dimensional dependent variables space. Let $x = (x^1, \dots, x^n) \in X$ and $u = (u^1, \dots, u^m) \in U$ with $u^j \in U^j$. We define the Jet-space $U^{(s)}$ as $U^{(s)} := \bigotimes_{j=1}^m \left(\bigotimes_{l=0}^s U_{(l)}^j \right)$, where $U_{(l)}^j$ is the set of all $p_l \equiv \binom{n+l-1}{l}$ distinct l -th order partial derivatives of u^j . We denote by $u_{(k)}^j$ the p_k -tuple of all k -order derivatives of u^j . An element $u^{(s)}$ in the Jet-space $U^{(s)}$ is the $m(1 + p_1 + p_2 + \dots + p_s) = m \binom{n+s}{s}$ -tuple defined by $u^{(s)} = (u_{(0)}^1, u_{(1)}^1, \dots, u_{(s)}^1, u_{(0)}^2, u_{(1)}^2, \dots, u_{(s)}^2, \dots, u_{(0)}^m, u_{(1)}^m, \dots, u_{(s)}^m)$.

A variational problem consists in finding extrema of a functional \mathfrak{L} defined by

$$\mathfrak{L}[u] = \int_{\Omega} L(x, u^{(s)}) dx, \quad (1)$$

where Ω is a connected open subset of X and L defined on $X \times U^{(s)}$ is an s -order differential function called the Lagrangian of the variational problem \mathfrak{L} . In general, a functional is a mapping that assigns to each element in some function space a real number, and a variational problem amounts to searching for functions which are an extremum (minimum, maximum, or saddle points) of a given functional.

Theorem 0.1. *Let u be an extremum of \mathfrak{L} , then u satisfies the Euler-Lagrange equations*

$$\frac{\delta}{\delta u^j} L(x, u^{(s)}) = 0, \quad j = 1, \dots, m. \quad (2)$$

Theorem 0.2. (Noether's theorem)

Let G be a one parameter variational symmetry group for the functional $\mathfrak{L}[u] = \int_{\Omega} L(x, u^{(s)}) dx$, and $V = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{j=1}^m \phi^j(x, u) \frac{\partial}{\partial u^j}$ be an infinitesimal generator of G , i.e. V satisfies the variational infinitesimal invariance criterion

$$Pr^{(s)} V(L) + L \operatorname{div} \xi = \operatorname{div} B \quad (3)$$

for some vector $B = (B^1, \dots, B^n)$ of differential functions, where $Pr^{(s)}X$ is the s -order prolongation of X . Then the vector $C = (C^1(x, u^{(s_1)}), \dots, C^n(x, u^{(s_n)}))$ defined by:

$$C^i = -B^i + \xi^i L + \sum_{j=1}^m \sum_{k_1=0}^{s_1^j} \dots \sum_{k_i=0}^{s_i^j-1} \dots \sum_{k_n=0}^{s_n^j} D_{x^1}^{k_1} \dots D_{x^n}^{k_n} (W^j) \\ \times \sum_{l_1=0}^{s_1^j-k_1} \dots \sum_{l_n=0}^{s_n^j-k_n} (-D_{x^1})^{l_1} \dots (-D_{x^n})^{l_n} \left(\frac{\partial L}{\partial u_{(k_1+l_1)x^1 \dots (k_i+l_i+1)x^i \dots (k_n+l_n)x^n}^j} \right)$$

where $W^j = \phi^j - \sum_{i=1}^n \xi^i u_{x^i}^j$, $j = 1, \dots, m$ provides a conservation law for the Euler-Lagrange equations (2): $\frac{\delta}{\delta u^j} L(x, u^{(s)}) = 0$, $j = 1, \dots, m$, i.e. obeys the equation $\text{div} C \equiv D_{x^1} C^1 + \dots + D_{x^n} C^n = 0$ for all solution of the system (2). Such a vector C is called a **conserved vector** for the system (2).

MAIN RESULTS FOR UNDER DETERMINED SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definition 0.1. (Adjoint equation)

Consider the system

$$F_\alpha \left(x, u^{(s)}, \tilde{u}^{(s)} \right) = 0, \quad \alpha = 1, \dots, m, \quad (4)$$

where F_α are differential functions having n independent variables $x = (x^1, \dots, x^n)$ and $m + \tilde{m}$ dependent variables $u = (u^1, \dots, u^m)$, $\tilde{u} = (\tilde{u}^1, \dots, \tilde{u}^{\tilde{m}})$, $u = u(x)$, $\tilde{u} = \tilde{u}(x)$; $u^{(s)}$ (resp. $\tilde{u}^{(s)}$) is a vector encompassing dependent variable u (resp. \tilde{u}) and their derivatives up to order s . We introduce the differential functions

$$F_\alpha^* = \frac{\delta}{\delta u^\alpha} \left[\sum_{\beta=1}^m v^\beta F_\beta + \left(\sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left(\sum_{v=1}^m F_v \right) \right] \\ \tilde{F}_\alpha^* = \frac{\delta}{\delta \tilde{u}^\alpha} \left[\sum_{\beta=1}^m v^\beta F_\beta + \left(\sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left(\sum_{v=1}^m F_v \right) \right],$$

where $v = (v^1, \dots, v^m)$ and $\tilde{v} = (\tilde{v}^1, \dots, \tilde{v}^{\tilde{m}})$ are new dependent variables, ($v = v(x)$, $\tilde{v} = \tilde{v}(x)$), also called nonlocal variables. Then, we define the corresponding system of adjoint equations by

$$F_\alpha^* \left(x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)} \right) = 0, \quad \alpha = 1, \dots, m \quad (5)$$

$$\tilde{F}_\alpha^* \left(x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)} \right) = 0, \quad \tilde{\alpha} = 1, \dots, \tilde{m}. \quad (6)$$

Theorem 0.3. Any system of PDEs (4): $F_\alpha(x, u^{(s)}, \tilde{u}^{(s)}) = 0$, $\alpha = 1, \dots, m$, considered together with their adjoint equations (5)-(6), has a Lagrangian. Namely, the Eqs (4)-(6) with $2(m + \tilde{m})$ unknowns are the Euler-Lagrange equations with Lagrangian

$$L(x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)}) = \sum_{\beta=1}^m v^\beta F_\beta + \left(\sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left(\sum_{\alpha=1}^m F_\alpha \right).$$

Proof. It is immediate from the definitions of Euler-Lagrange equations and adjoint equations. ■

Definition 0.2. The system (4) is called self-adjoint if the substitution $(v, \tilde{v}) = (u, \tilde{u})$ in its adjoint Eqs. (5)-(6) gives, for some differential functions $\Gamma_{\alpha v}$ and $\tilde{\Gamma}_{\tilde{\alpha} \tilde{v}}$, $F_\alpha^* = \sum_{v=1}^m \Gamma_{\alpha v} F_v$, $\alpha = 1, \dots, m$, and $\tilde{F}_{\tilde{\alpha}}^* = \sum_{\tilde{v}=1}^{\tilde{m}} \tilde{\Gamma}_{\tilde{\alpha} \tilde{v}} F_{\tilde{v}}$, $\tilde{\alpha} = 1, \dots, \tilde{m}$. The system (4) is called quasi-self-adjoint if there exist two functions h and \tilde{h} such that the same expansions of F_α^* and $\tilde{F}_{\tilde{\alpha}}^*$ in terms of $\Gamma_{\alpha v}$ and $\tilde{\Gamma}_{\tilde{\alpha} \tilde{v}}$ hold upon the substitution $(v, \tilde{v}) = (h(u, \tilde{u}), \tilde{h}(u, \tilde{u}))$.

Provided these statements, we can now provide the main results of this paper.

Theorem 0.4. Consider the system (4). Then Its adjoint Eqs. (5)-(6) inherits its symmetries of equations (4). Namely, if the system (4) admits an operator $X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_\beta \frac{\partial}{\partial u^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}} \frac{\partial}{\partial \tilde{u}^{\tilde{\beta}}}$, where X is a generator of a point transformation group, i.e. $\xi^i = \xi^i(x, u, \tilde{u})$, $\eta_\beta = \eta_\beta(x, u, \tilde{u})$, $\tilde{\eta}_{\tilde{\beta}} = \tilde{\eta}_{\tilde{\beta}}(x, u, \tilde{u})$ and $Pr^{(s)}X(F_\alpha) = \sum_{\beta=1}^m \lambda_{\alpha\beta}(x, u^{(s)}, \tilde{u}^{(s)}) F_\beta$, then the equations (5)-(6) have the generator of symmetries $Y = X + \sum_{\beta=1}^m \eta_\beta^* \frac{\partial}{\partial v^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}}^* \frac{\partial}{\partial \tilde{v}^{\tilde{\beta}}}$, where $\eta_\beta^* + \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\alpha}}^* = - \left[\sum_{\alpha=1}^m v^\alpha \lambda_{\alpha\beta} + v^\beta \sum_{i=1}^n D_{x^i}(\xi^i) + \left(\sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \sum_{\alpha=1}^m \lambda_{\alpha\beta} + \left(\sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \left(\sum_{i=1}^n D_{x^i}(\xi^i) \right) \right]$, which is satisfied in particular for $\eta_\beta^* = - \left[\sum_{\alpha=1}^m \left(v^\alpha + \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \lambda_{\alpha\beta} + v^\beta \sum_{i=1}^n D_{x^i}(\xi^i) \right]$, $\tilde{\eta}_{\tilde{\alpha}}^* = - \tilde{v}^{\tilde{\alpha}} \sum_{i=1}^n D_{x^i}(\xi^i)$.

Proof. Using the variational infinitesimal test and setting the coefficients of F_β to 0 yield the results. ■

Theorem 0.5. Consider the system (4) Its adjoint Eqs. (5)-(6) inherits Lie-Bäcklund operator of equations (4). Namely, if the system (4) admits an operator $X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_\beta \frac{\partial}{\partial u^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}} \frac{\partial}{\partial \tilde{u}^{\tilde{\beta}}}$, where X is a Lie-Bäcklund operator, i.e. $\xi^i = \xi^i(x, u^{(s_1)}, \tilde{u}^{(s_1)})$, $\eta_\beta = \eta_\beta(x, u^{(s_2)}, \tilde{u}^{(s_2)})$, $\tilde{\eta}_{\tilde{\beta}} = \tilde{\eta}_{\tilde{\beta}}(x, u^{(s_3)}, \tilde{u}^{(s_3)})$ are any differential functions, and $PrX(F_v) = \sum_{\mu=1}^m \mathcal{D}_{v\mu}(F_\mu)$ for some differential operators $\mathcal{D}_{v\mu} = \lambda_{v\mu}^0 + \sum_{i_1=1}^n \lambda_{v\mu}^{i_1} D_{x^{i_1}} + \sum_{i_1, i_2=1}^n \lambda_{v\mu}^{i_1 i_2} D_{x^{i_1}} D_{x^{i_2}} + \sum_{i_1, i_2, i_3=1}^n \lambda_{v\mu}^{i_1 i_2 i_3} D_{x^{i_1}} D_{x^{i_2}} D_{x^{i_3}} + \dots$, then equations (5)-(6) admit the Lie-Bäcklund operator $Y = X + \sum_{\beta=1}^m \eta_\beta^* \frac{\partial}{\partial v^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}}^* \frac{\partial}{\partial \tilde{v}^{\tilde{\beta}}}$,

where

$$\eta_{\beta}^* + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}}^* = - \left\{ w^{\beta} \sum_{i_1=1}^n D_{x^{i_1}} (\xi^{i_1}) + \sum_{\mu=1}^m \left[w^{\mu} \lambda_{\mu\beta}^0 - \sum_{i_1=1}^n D_{x^{i_1}} (w^{\mu} \lambda_{\mu\beta}^{i_1}) \right. \right. \\ \left. \left. + \sum_{i_1, i_2=1}^n D_{x^{i_1}} D_{x^{i_2}} (w^{\mu} \lambda_{\mu\beta}^{i_1 i_2}) - \sum_{i_1, i_2, i_3=1}^n D_{x^{i_1}} D_{x^{i_2}} D_{x^{i_3}} (w^{\mu} \lambda_{\mu\beta}^{i_1 i_2 i_3}) + \dots \right] \right\}$$

with $w^{\alpha} = v^{\alpha} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}}$.

Proof. Replacing $w^{\beta} = v^{\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}}$ in the result of the computation of $T \equiv \text{Pr}Y(L) + L\text{Div}(\xi)$ and setting the coefficients of F_{β} to 0, then the resulting expression of T reduces to a divergence, what achieves the proof. ■

Finally, we arrive at the following general formulation of the conservation theorem.

Theorem 0.6. *Every infinitesimal generator (or Lie-Bäcklund operator)*

$$X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_{\beta} \frac{\partial}{\partial u^{\beta}} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}} \frac{\partial}{\partial \tilde{u}^{\tilde{\beta}}} \quad (7)$$

of differential equations (4) provides a conservation law for the system of differential equations (4) and their adjoint (5)-(6).

Relevant nonlinear partial differential equations of mathematical physics such as the three dimensional time-dependent Ginzburg-Landau (GL) equations [20, 21] are studied in this framework. Main results of these investigations will be in the core of forthcoming papers.

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REFERENCES

1. R. Naz, F. M. Mahomed and D. P. Mason, Comparison of different approaches to conservation laws for some partial differential equations in fluids mechanics, *Appl. Math. Comput.* **205** (2008), pp. 212-230.
2. V. Rosenhaus, On conserved densities and asymptotic behaviour for the potential Kadomtsev-Petviashvili equation, *J. Phys. A: Math. Gen.* **39** (2006), pp. 7693-7703.
3. J. M. Sanz-Serna, An explicit finite-difference scheme with exact conservation properties, *J. Comput. Phys.* **47** (1982), pp. 199-210.
4. T. Wolf, Application of CRACK in the classification of integrable systems, *CRM Proceedings and Lectures Notes* **37** (2004), pp. 283-300.
5. T. Wolf, Partial and complete linearization of PDEs based on conservation laws, *Trends in Mathematics: Differential Equations with Symbolic Computation* (2005), pp. 291-306.

6. C. G. J. Jacobi, Vorlesungen ber Dynamik, 2nd ed., Reimer, Berlin, 1884.
7. Klein, F. Über die differentialgesetze für die Erhaltung von Impuls und Energie in der Einsteinschen Gravittionstheorie, Nachr. König. Gesell. Wissen, Göttingen, Math.Phys. Kl., Heft 2, pp. 171-189, 1918.
8. E. Noether, Invariant Variation problems, *Nachr. v. d. Ges. d. Wiss. zu Göttingen* (1918), pp. 235-257.
9. Bessel-Hagen Mathem. Ann., 84, pp. 258-276, 1921. English trans. in Archives of ALGA, 3, ALGA publ., BTH, Karlskrona, Sweden, (2006), pp. 33-51.
10. Hill, Rev. Mod. Phys., 23, (1951), pp. 253-260.
11. N. H. Ibragimov, Teoreticheskaya i Matematicheskaya Fizika, 1(3), (1969), pp. 350-359. English trans. in Theor. Math. Phys., 1(3), (1969), pp. 267-276.
12. T. Wolf, A comparison of four appoaches to the calculation of conservation laws, *Euro. J. Appl. Math.* **13** (2002), pp. 129-152.
13. S. C. Anco and G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation laws classifications, *Europ. J. Appl. Math.* **13** (2002), pp. 545-566.
14. S. C. Anco and G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part II: General treatment, *Europ. J. Appl. Math.* **13** (2002), pp. 567-585.
15. L. D. Poole, Symbolic computation of conservation laws of nonlinear partial differential equations using homotopy operators, *PhD Thesis, Colodaro School of Mines* (2007).
16. N. H. Ibragimov, Integrating factors, adjoint equations and Lagrangians, *J. Math. Anal. Appl.*, 318 (2), (2006).
17. N. H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.* **333** (1) (2007).
18. R. Naz, Symmetry solutions and conservation laws for some partial differential equations in field mechanics, PhD Thesis, Växjö University, Sweden (2008).
19. R. Khamitova, Symmetries and conservation laws, PhD Thesis, University of the Witsatersrand, Johannesburg, (2009).
20. D. Qiang, Numerical approximation of Ginzburg-Landau for superconductivity, *J. Math. Phys* **46** (2005).
21. D. Qiang, Global existence and uniqueness of solution of the time-dependent Ginzburg-Landau equations in superconductivity, *Appl. Anal.* **52** (1994), pp. 1-17.